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Supersymmetric Models for Fermions on a Lattice

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We dedicate this article to Julius Wess on the occasion of his 70^{th} *birthday.*

Abstract

We investigate the large-N behaviour of simple examples of supersymmetric interactions for fermions on a lattice. Witten's supersymmetric quantum mechanics and the BCS model appear just as two different aspects of one and the same model. For the BCS model, supersymmetry is only respected in a coherent superposition of Bogoliubov states. In this coherent superposition mesoscopic observables show better stability properties than in a Bogoliubov state.

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1 Introduction

We realize the simplest supersymmetric system by a finite fermion lattice. The supersymmetric structure is determined by a non-hermitian supercharge Q. This is an odd nilpotent element of the Fermi algebra $\mathcal{A}$. The sum and the anticommutator of Q with its adjoint $Q^\dagger$ give two hermitian elements, G and $H = G^2$. They generate the supertransformation and the time evolution, two commuting automorphism groups of $\mathcal{A}$. Already at this level of generality the Hilbert space assumes a structure. It is the sum of the null-space of these operators plus the tensor product of $\mathcal{C}^2$ and a rest $\mathcal{H}_H$. In $\mathcal{H}_H$ H has strictly positive eigenvalues, which are twofold degenerate, while G has the eigenvalues $\pm\sqrt{H}$ and leaves these spaces invariant. Thus the supertransformation and the time evolution are linked. However this closeness is lost once we go to the limit where N, the number of lattice points, goes to infinity. There it can even happen that on local operators the supertransformation is not well defined whereas the time evolution is.

To restrict the many possibilities for Q, we impose on it a locality condition: products of operators may only contain operators of the same lattice site. If there is only one fermion per site, G and H are essentially unique. The time evolution becomes trivial, but the supertransformation remains and is in fact non-local. If there are two fermions per site, the time evolution becomes the free time evolution and there is a non-local supertransformation associated to it.

Since even for a local Q the supertransformation is non-local, we drop the locality requirement. We construct a Clifford variable η, which is usually associated with supersymmetry. Note that a Grassman η would mean $\{\eta, \eta^\dagger\} = 0$, which is not possible in a $\mathcal{C}^*$-algebra, but our η is nilpotent and anticommutes with the other odd elements. With three fermions per site we can construct a supersymmmetric version of the BCS model. This is mathematically well explored and we can behold the many vistas that the limit $N \to \infty$ offers. The limit depends on the state on which the representation is based and we shall study it for three different states: the ground state of H, its ceiling state, and the Bogoliubov state, which in the limit $N \to \infty$ has the same energy per particle as the ceiling state but remains pure on the quasi-local algebra.

We shall study the following three types of limiting observables:

a) Local operators, i.e. polynomials in the operators A_i, the elements of $\mathcal{A}$, which are localized at the site i;

b) Mesoscopic observables, i.e. limits of $\frac{1}{\sqrt{N}} \sum_{i=1}^N A_i$;

c) Macroscopic observables, i.e. limits of $\frac{1}{N} \sum_{i=1}^N A_i$.

Our automorphisms turn out to be different in all cases. It is not even true that the microscopic time evolution determines the mesoscopic one. The supertransformation is finite and non-trivial only for the ground state, where Witten's supersymmetric quantum mechanics emerges in the limiting procedure. It seems remarkable that the difference between a statistical mixture of the Bogoliubov states and the ceiling state, which can be considered as a coherent mixture of the Bogoliubov states, can only be observed on the mesoscopic and macroscopic level respectively. Especially the mesoscopic algebra is stable in the ceiling state under the emerging time evolution, whereas it is not so in the

Bogoliubov state, which also breaks supersymmetry.

2 Algebraic framework

The basic structural elements of supersymmetry are a C^*-algebra $\mathcal{A}$ and an odd nilpotent element $Q \in \mathcal{A}$ (the supercharge):

$$Q^2 = 0 \Rightarrow \{Q^\dagger\}^2 = 0 \qquad QQ^\dagger + Q^\dagger Q =: H. \tag{2.1}$$

H is supposed to be the generator of the time evolution and by Eq.(2.1) has the properties

$$\begin{aligned} (i) &\quad [Q,H] = 0 \Leftrightarrow [Q^\dagger, H] = 0 \\ (ii) &\quad H|0\rangle = 0 \Leftrightarrow Q|0\rangle = Q^\dagger|0\rangle = 0 \\ (iii) &\quad E:\ H|e\rangle = E|e\rangle,\ E > 0 \Rightarrow \qquad \text{is at least twofold degenerate.} \end{aligned} \tag{2.2}$$

For *(iii)*, note that either $Q|e\rangle$ or $Q^\dagger|e\rangle$ must be different from zero and also belongs to the eigenvalue E. Also, $Q|e\rangle$ cannot be $\sim |e\rangle$, $Q^2|e\rangle = 0$ would be in contradiction with the assumption $Q|e\rangle \neq 0$.

Equation (2.2) implies that the Hilbert space $\mathcal{H}$ can be written as a sum of a zero-space $\mathcal{H}_0$ (projection $P_0 : \mathcal{H}_0 = P_0\mathcal{H}$) and a tensor product of $\mathcal{C}^2$ and the rest, $\mathcal{H}_H$: $\mathcal{H} = \mathcal{H}_0 \bigoplus (\mathcal{C}^2 \otimes \mathcal{H}_H)$. Defining $\eta = Q/\sqrt{H}, P_0\eta = 0$ we have $\eta\eta^\dagger + \eta^\dagger\eta = 1 - P_0$. In this decomposition we can write η, H, Q in a matrix representation

$$\eta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & H & 0 \\ 0 & 0 & H \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{H} \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.3}$$

Here we identify $H = H(1 - P_0)$; $\sqrt{H}$ denotes the positive square root of $H \in \mathcal{B}(\mathcal{H}_H)$, but there are others: if $G_\alpha = e^{i\alpha}Q + e^{-i\alpha}Q^\dagger, \alpha \in (0, 2\pi)$, then $G_\alpha^2 = H\,\forall\alpha$. The gauge transformation $G = G_0 \to G_\alpha$ is effected by $F = [\eta, \eta^\dagger]$, which has the familiar matrix representation

$$F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad G = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{H} \\ 0 & \sqrt{H} & 0 \end{pmatrix}, \tag{2.4}$$

so that $e^{i\alpha F} G e^{-i\alpha F} = G_\alpha$.

In $\mathcal{C}^2 \otimes \mathcal{H}_H$ every element can be written as $A\eta\eta^\dagger + B\eta^\dagger\eta + C\eta + D\eta^\dagger, A, B, C, D \in \mathcal{B}(\mathcal{H}_H)$. This gives the algebra a grading by calling the first two terms even and the others odd. All this emerges from a single nilpotent operator, namely Q. The operators G_α generate a supertransformation

$$a_k \to a_k(s, \alpha) = e^{isG_\alpha} a_k e^{-isG_\alpha}, \tag{2.5}$$

which mixes even and odd elements of $\mathcal{A}$. To isolate the various aspects, we start with a finite-dimensional $\mathcal{A}$, $i, k = 1, 2, ..., N$ and later investigate the limit $N \to \infty$.

3 Fermions on a lattice

The supertransformation is a non-linear transformation of the a's that preserves their algebraic relations. For warming up we start with the simplest case:

$$N = 1,\ Q = a,\ G = e^{i\alpha} a + e^{-i\alpha} a^\dagger .$$

The supertransformation is

$$a(s, \alpha) = (\cos s)^2 a + e^{2i\alpha} (\sin s)^2 a^\dagger + i e^{i\alpha} \cos s \sin s (a^\dagger a - a a^\dagger). \tag{3.1}$$

In this special case H has to be trivial since it is twofold degenerate, *(ii)* cannot appear since $a, a^\dagger$ create all of $\mathcal{A}$. The supertransformation $a \to a(s, \alpha)$ is a two-dimensional generating subset (but not subgroup) of the automorphism group. The latter is three-dimensional and isomorphic to $SU(2)$. Equation (2.3) tells us that in its embryonic form the supertransformation is a Bogoliubov transformation plus a quadratic term.

To get such an explicit expression also for higher N we restrict the systems to be considered first by a locality condition. We think in terms of a lattice and assume Q to be the sum of charges situated at the lattice sites:

$$Q = \sum_{i=1}^{N} q_i . \tag{3.2}$$

Equation (2.1) requires $\{q_i, q_j^\dagger\} = 0\ \forall i \neq j$, in addition to $\{q_i, q_j\} = 0$.

<u>**I.** One kind of fermions at each lattice site</u>

The most general Q is of the form

$$Q = \sum_i z_i a_i,\ \ z_i \in \mathbf{C}. \tag{3.3}$$

Since the third power of a fermion operator at a point vanishes, q_i must be linear in a and $a^\dagger$, and the nilpotence leaves only a or $a^\dagger$ (which one does not matter). The phase of the a_i is arbitrary, so we may take $z_i \in \mathbf{R}^+$, that is $\alpha_i = 0$. Thus

$$G = \sum_i z_i (a_i + a_i^\dagger)\,, \quad H = \sum_i z_i^2 , \tag{3.4}$$

so with H being a c-number the time evolution is trivial. The $\mathcal{H}_0$ introduced in Section 2 is empty, a vector $|0\rangle$ with $Q|0\rangle = Q^\dagger|0\rangle = 0$ would be annihilated by all a_i and $a_i^\dagger$. Since they span all of $\mathcal{A}$ such a vector must be zero. The η of Section 2 is $\sum_i z_i a_i / (\sum_k z_k)^{-1/2}$ and is a collective fermion coordinate. Since it obeys the CAR-relations we have $||\eta|| = 1$ although it is a sum of N operators with of a norm $N^{-1/2}$.

However, the supertransformation e^{isG} is not trivial; also it is not just a tensor product of the unitaries of the baby model since the q_i and $q_i^\dagger$ in (3.2) at different points anticommute. We find rather

$$G a_k = z_k - a_k G \ \to\ G^n a_k = \frac{1 - (-1)^n}{2} z_k G^{n-1} + (-1)^n a_k G^n$$

$$\Rightarrow\ e^{isG}a_k = a_k e^{-isG} + z_k\,\frac{e^{isG}-e^{-isG}}{2G}\ ,$$

so that

$$a_k(s) = e^{isG}a_k e^{-isG} = (a_k - z_k/2G)\,e^{-2isG} + z_k/2G\ , \tag{3.5}$$

where we use the notation $a_k(0) = a_k$. One readily verifies that this is an automorphism of $\mathcal{A}$,

$$a_k^2(s) = 0\,, \quad \{a_k(s), a_j^\dagger(s)\} = \delta_{kj}\,, \quad \{a_k(s), a_j(s)\} = 0$$

but it is not a local transformation — $a_k(s)$ depend on all the other a's and $a^\dagger$'s.

II. Two fermions at each lattice site

We think of it as of electrons with spin up and down. Thus $\mathcal{A}$ is generated by $a_{\uparrow,i}$ and $a_{\downarrow,i}$. So now we afford at each site a product of three operators, of which there are four types:

$$a_\uparrow^\dagger a_\uparrow a_\downarrow\ ,\ a_\uparrow^\dagger a_\downarrow a_\downarrow^\dagger\ ,\ a_\uparrow a_\downarrow^\dagger a_\downarrow\ ,\ a_\uparrow a_\uparrow^\dagger a_\downarrow^\dagger\ .$$

Each of them is nilpotent. A typical local supercharge is

$$Q = \sum_i a_{\uparrow,i}^\dagger a_{\uparrow,i} a_{\downarrow,i}\, z_i\ ,\ \ z_i \in \mathbf{R}^+ \tag{3.6}$$

and

$$G = \sum_i z_i a_{\uparrow,i}^\dagger a_{\uparrow,i}(a_{\downarrow,i} + a_{\downarrow,i}^\dagger). \tag{3.7}$$

A priory, $H = G^2$ seems to be of 6-th power in the a's but by locality it can be only quartic and $(a^\dagger a)^2 = a^\dagger a$ reduces it to something quadratic:

$$H = \sum_i z_i^2\, a_{\uparrow,i}^\dagger a_{\uparrow,i}. \tag{3.8}$$

Thus a free time evolution where half of the fermions are quiet is supersymmetric with a local supercharge. In this case the vacuum $|\uparrow 0\rangle, a_{\uparrow,i}|\uparrow 0\rangle$ satisfies

$$Q|\uparrow 0\rangle = Q^\dagger|\uparrow 0\rangle = 0,$$

irrespective of the down spins. All these vectors belong to the eigenvalue zero of H and span $\mathcal{H}_0$. In fact the eigenvalues are at least 2^N-fold degenerate.

We determine the automorphism of $\mathcal{A}$ generated by the supertransformation e^{isG} by the same method as before:

$$Ga_{\uparrow,k} = z_k a_{\uparrow,k}(a_{\downarrow,k} + a_{\downarrow,k}^\dagger) - a_{\uparrow,k}G\ \to\ G^n a_{\uparrow,k} = (-1)^n a_{\uparrow,k}(G - z_k(a_{\downarrow,k} + a_{\downarrow,k}^\dagger))^n,$$

leading to

$$a_{\uparrow,k}(s) = a_{\uparrow,k}e^{-is(G-(a_{\downarrow,k}+a_{\downarrow,k}^\dagger)z_k)}e^{-isG}. \tag{3.9}$$

Similarly,

$$Ga_{\downarrow,k} = z_k a_{\uparrow,k}^\dagger a_{\uparrow,k} - a_{\downarrow,k}G\ \to\ G^n a_{\downarrow,k} = (-1)^n a_{\downarrow,k}G^n + \frac{1-(-1)^n}{2} z_k a_{\uparrow,k}^\dagger a_{\uparrow,k} G^{n-1},$$

leading to

$$a_{\downarrow,k}(s) = a_{\downarrow,k}e^{-2isG} + z_k a^\dagger_{\uparrow,k} a_{\uparrow,k} \frac{1-e^{-2isG}}{2G}. \tag{3.10}$$

Of course there is an alternative form where e^{2isG} is pulled out to the left and with which one verifies that this strange transformation $a_{\uparrow\downarrow,k} \to a_{\uparrow\downarrow,k}(s)$ is actually an automorphism group of $\mathcal{A}$ that mixes spin up and down as well as even and odd; however, the time evolution leaves $a_{\downarrow,k}$ invariant up to a phase and does not mix between even and odd elements or between elements at different sites.

Remark
Instead of an opposite spin one might use the next neighbour and try

$$Q = \sum_i a^\dagger_i a_i a_{i+1} z_i, \tag{3.11}$$

but this is not nilpotent. To meet this condition more refined constructions are necessary (see, e.g. [1]).

III. Three fermions at each lattice site

In the case of three fermions at each lattice site, we start again with the CAR-algebra $\mathcal{A}$ generated by $\{a^\alpha_i\}$:

$$\{a^\alpha_i, a^{\dagger\,\beta}_k\} = \delta^{\alpha\beta}\delta_{ik},\ \{a^\alpha_i, a^\beta_k\} = 0, \qquad i,k = 1,\dots,N,\ \alpha,\beta = 1,2,3. \tag{3.12}$$

However even with strictly local q_i the charge $Q = \sum_i^N q_i$ in (3.2) creates a non-local supertransformation, so we drop locality. Instead we impose translation invariance of the q_i's in such a way that Q becomes translation- and even permutation- invariant. Furthermore we think of the a^1_i and a^2_i as Cooper pairs, and thus consider the subalgebra $\mathcal{C}$ of $\mathcal{A}$ generated by $b_i = a^1_i a^2_i$ and a^3_k, $i,k = 1,\dots,N$. Although the b_i's commute for different sites they do not form a *bona fide* Bose field since there is at most one pair per site, $b^2 = 0$. However in $\mathcal{C}$ the anticommutator $\{b^\dagger_i, b_i\}$ is a projection of the centre and therefore in an irreducible representation it equals unity. These are the representations we are interested in and therefore we can think of the b's as of spin variables:

$$b_i = \frac{\sigma^x_i - i\sigma^y_i}{2}, \qquad 1 - 2b^\dagger_i b_i = \sigma^z_i.$$

Thus our algebra $\mathcal{C}$ is defined by

$$\begin{aligned} &\{a^3_i, a^{3\dagger}_j\} = \delta_{ij},\ \{a^3_i, a^3_j\} = 0,\ [a^3_i, b_k] = 0 \\ &[b_i, b^\dagger_k] = \delta_{ik}(1 - 2b^\dagger_i b_i), \quad \{b_i, b^\dagger_i\} = 1. \end{aligned} \tag{3.13}$$

The supertransformation, and therefore the dynamics, will be defined by fluctuation variables.

Definition

$$M_N = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} b_k, \qquad \eta_N = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} a_k^3. \tag{3.14}$$

Proposition

(i) $\eta_N \eta_N^\dagger + \eta_N^\dagger \eta_N = 1, \quad \eta_N^2 = 0;$

(ii) $[M_N, M_N^\dagger] = 1 - \frac{2}{N} \sum_{k=1}^{N} b_k^\dagger b_k;$

(iii) $[M_N, \eta_N] = [M_N, \eta_N^\dagger] = 0.$

Remarks

1. η represents a collective Fermi mode and will serve as a Clifford variable. The fact that the number of single fermion modes s equals the number of pairs is not essential;

2. M is a collective Bose mode and, in a representation based on the "vacuum" $|0\rangle$: $b_i|0\rangle = 0$ (all spins down), it assumes for $N \to \infty$ the properties of $(x + ip)/\sqrt{2}$ in quantum mechanics and we will arrive at Witten's supersymmetric quantum mechanics [2];

3. By anticommutativity the a_k^3 are so correlated that $\|\eta_N\| = 1 \ \forall N$. On the contrary, $\|M_N\| = \sqrt{N}/2$ since we can think of M_N as of $(S^x - iS^y)/(2\sqrt{N})$, with $S = \sum_{k=1}^{N} \sigma_k$, $b = (\sigma^x - i\sigma^y)/2$.

In agreement with our desideratum $Q = \sum_i q_i$, $\{q_i, q_j\} = 0$, we take $q_i = b_i \eta/\sqrt{N}$ and thus obtain

$$\begin{aligned} &Q_N = M_N \eta_N \\ &G = Q_N + Q_N^\dagger \\ &H_{SS} = G^2 = \{Q_N, Q_N^\dagger\} = \\ &\quad = \eta_N \eta_N^\dagger M_N M_N^\dagger + \eta_N^\dagger \eta_N M_N^\dagger M_N = M_N^\dagger M_N + \eta_N \eta_N^\dagger \left(1 - \frac{2}{N} \sum_{k=1}^{N} b_k^\dagger b_k\right). \end{aligned} \tag{3.15}$$

Remarks

1. In the BCS model (in the degenerated case) $H_{\mathrm{BCS}} = -M_N^\dagger M_N$ and we see that it differs from $-H_{SS}$ only by $O(1)$. Thus the energies per particle H/N coincide for $N \to \infty$;

2. In the $S_z = 0$ situation, where $M_N \overset{N\to\infty}{\to} (x+ip)/\sqrt{2}$ with Pauli matrices for $\eta = (\tau_x + i\tau_y)/2$, we have $H_{SS} = (x^2+p^2)/2 + \tau_z/2$.

Next we shall briefly comment on the ground state and the ceiling state of H_{SS}. For their discussion it has to be kept in mind that, for $N \to \infty$, η_N stays bounded and only M_N can grow big; H_{SS} is then essentially $(S^x)^2 + (S^y)^2 = S^2 - (S^z)^2$. The smallest H_{SS} requires S^z as big as S. But the lowest $H_{\mathrm{BCS}} = -H_{\mathrm{SS}}$ wants S^z close to zero and maximal S. The Fock vacuum $b|0\rangle = 0$ gives $H_{\mathrm{SS}} = 0$ if the single fermions are anticorrelated to the pairs, $\eta^\dagger|0\rangle = 0$. Then $G|0\rangle = (M_N\eta_N + M_N^\dagger\eta_N^\dagger)|0\rangle = 0$ and thus $Q_N|0\rangle = Q_N^\dagger|0\rangle = H_{\mathrm{SS}}|0\rangle = 0$. For even N there are more ground states if all pairs are pair-wise anticorrelated. By this we mean that there is a permutation p_i, $i = 1, \ldots, N/2$ such that $b_i|0\rangle = -b_{i+p_i}|0\rangle$ and $b_i^\dagger|0\rangle = -b_{i+p_i}|0\rangle$. Then $M_N|0\rangle = M_N^\dagger|0\rangle = 0$ and again $Q_N|0\rangle = Q_N^\dagger|0\rangle = 0$, however η_N acts on $|0\rangle$. In this case $S = S^z = 0$.

For the ground state of $-H_{\mathrm{SS}}$ we want $S^z = 0$, $S = N$. For even N this is possible by applying $N/2$ times $M_N^\dagger$ onto $|0\rangle$. For big N this becomes awkward and here it is expedient to make a Bogoliubov transformation. However the standard form

$$a^1 \to \frac{a^1 + a^{2\dagger}}{\sqrt{2}}, \qquad a^2 \to \frac{a^2 - a^{1\dagger}}{\sqrt{2}},$$

does not leave the pair algebra $\mathcal{C} \subset \mathcal{A}$ invariant and we have to use the transformation (2.1) with $\cos s = 0$, $\sin s = 1$, i.e.

$$b \to \frac{b - b^\dagger}{2} + \frac{1}{2} - b^\dagger b,$$

such that M_N becomes

$$M_N = \frac{\sqrt{N}}{2} + \frac{1}{\sqrt{N}} \sum_i \left[b_i^\dagger b_i + \frac{1}{2}(b_i - b_i^\dagger) \right].$$

If $|0\rangle$ denotes the new b-vacuum we note $||M_N - \frac{\sqrt{N}}{2}|0\rangle||^2 = 1/4$. We conclude

$$\lim_{N\to\infty} ||\frac{4}{N} M_N^\dagger M_N|| = 1. \tag{3.16}$$

4 The limit $N \to \infty$

In the limit $N \to \infty$, new features appear. We have to distinguish between local, mesoscopic and macroscopic observables: typically $\vec{\sigma}^j, \lim_{N\to\infty} \frac{\vec{S}}{\sqrt{N}}, \lim_{N\to\infty} \frac{\vec{S}}{N}$. Which limits exist and how they behave under the time evolution (TE) and the supertransformation (ST) will depend very much on the state.

For the limiting procedure we impose only minimal requirements. We assume that a state for arbitrary N is given and we check whether the expectation values converge.

In addition we demand that these limits can be interpreted as the expectation values of a limiting algebra. We want the latter to be as big as necessary for the mesoscopic observables to still reflect some quantum features.

Let us first turn to the global unscaled quantities

$$
\begin{aligned}
&S_\alpha = \sum_{k=1}^{N} \sigma_\alpha^k && S_\pm = \frac{1}{2}(S_x \pm iS_y) \\
&[S_+, S_-] = S_z && [S_z, S_\pm] = \pm 2S_\pm .
\end{aligned}
\tag{4.1}
$$

In terms of these operators together with η from (3.14) in the model of Section 3:

$$
\begin{aligned}
G_\alpha &= \frac{1}{\sqrt{N}}(e^{i\alpha}\eta S_- + e^{-i\alpha}\eta^\dagger S_+) \\
H_{\mathrm{SS}} = G_\alpha^2 &= \frac{1}{N}\{S_+S_- + [S_-,\, S_+]\eta\eta^\dagger\}
\end{aligned}
\tag{4.2}
$$

H_{SS} is independent of the gauge transformation $\gamma_\alpha G = G_\alpha$. For the time evolution (with $\dot{A} = -i[A, H]$) this leads to

$$
\begin{aligned}
\dot{\sigma}_z^{(j)} &= -i\sigma_+^{(j)}\frac{S_-}{N} + i\frac{S_+}{N}\sigma_-^{(j)} \\
\dot{\sigma}_+^{(j)} &= -i\frac{S_+}{N}\sigma_z^{(j)} - i\frac{2\sigma_+^{(j)}\eta\eta^\dagger}{N} \\
\frac{\dot{S}_+}{N} &= -i\frac{S_+S_z}{N^2} - \frac{2S_+}{N^2}\eta\eta^\dagger \qquad \dot{\eta} = -i\frac{\eta}{N}[S_-, S_+] \qquad \dot{S}_z = 0,
\end{aligned}
\tag{4.3}
$$

whereas for the supertransformation ($A' = -i[A, G_\alpha]$) we obtain

$$
\begin{aligned}
\sigma_z^{(j)\prime} &= 2ie^{i\alpha}\eta\frac{\sigma_-}{\sqrt{N}} - 2ie^{-i\alpha}\eta^\dagger\frac{\sigma_+}{\sqrt{N}} \\
\sigma_+^{(j)\prime} &= -ie^{i\alpha}\eta\frac{\sigma_j^{(j)}}{\sqrt{N}} \\
\frac{S_+'}{N} &= ie^{i\alpha}\eta\frac{S_z}{N\sqrt{N}} \qquad \eta' = i\frac{S_+}{\sqrt{N}}[\eta, \eta^\dagger] \\
\frac{S_z'}{N} &= 2i\frac{1}{N\sqrt{N}}(e^{i\alpha}\eta S_- - e^{-i\alpha}\eta^\dagger S_+).
\end{aligned}
\tag{4.4}
$$

For $N \to \infty$ evidently $\lim \vec{\sigma}^{(j)\prime} = 0$. For the time evolution we can use the fact that $\vec{S}/N$ is a norm bounded sequence. Therefore if $N \to \infty$ the time derivatives will have weak accumulation points. To be able to construct a corresponding automorphism group, however, we need strong convergence that will only hold in favourable representations. Especially $\vec{S}/N$ will be in the centre of the representation, and supersymmetry becomes trivial in local and global operators $\vec{\sigma}$ and $\vec{S}/N$.

The ground state of $H_{\rm SS}$

The ground state is given as the expectation value with the "vacuum vector" $|0\rangle$ with all spins down: $S_z^{(N)}|0\rangle = -N|0\rangle$, $(\overrightarrow{S}^{(N)})^2|0\rangle = N(N+2)|0\rangle$. This means for quasi-local operators:

$$\omega(\Pi_j \sigma^j_{k_j}) = \Pi_j(-\delta_{k_j,3}). \tag{4.5}$$

The expectation value factorizes and is the same for all k. Following [3] we can extend the set of observables and introduce the fluctuation operators

$$W_N(\alpha,\beta) := \exp\left\{ i \sum_{k=1}^{N} \frac{\alpha\sigma_x^k - \beta\sigma_y^k}{\sqrt{2}N} \right\}. \tag{4.6}$$

Hence

$$\begin{aligned} \langle 0|W_N(\alpha,\beta)|0\rangle &= \langle 0| \downarrow \ldots \downarrow |1 - \tfrac{\alpha^2+\beta^2}{4N} + O(N^{-2})| \downarrow \ldots \downarrow\rangle^N \\ &\longrightarrow e^{-(\alpha^2+\beta^2)/4} = \langle 0|e^{i(\alpha q+\beta p)}|0\rangle, \end{aligned}$$

where the vector $|0\rangle$ is now the ground state of a harmonic oscillator over a Weyl algebra $\mathcal{W}$ generated canonically by x and p. If in addition to $W_N(\alpha,\beta)$ we have a local polynomial $\prod_{j=1}^M \sigma^j_{\alpha_j}$, then for fixed M and $N \to \infty$ the expectation value factorizes since the interference between the two factors vanishes as $1/\sqrt{N}$.

Furthermore

$$\begin{aligned} \langle 0|W_N(\alpha,0)W_N(0,\beta)|0\rangle &= (\langle \downarrow \ldots \downarrow |(1 + i\tfrac{\alpha\sigma_x}{\sqrt{2}N} - \tfrac{\alpha^2}{4N})(1 - i\tfrac{\beta\sigma_x}{\sqrt{2}N} - \tfrac{\beta^2}{4N})| \downarrow \ldots \downarrow\rangle)^N \\ &\to (1 - \tfrac{\alpha^2+\beta^2}{4N} + i\tfrac{\alpha\beta}{2N})^N \to \langle 0|W(\alpha,\beta)|0\rangle e^{-i4\alpha\beta}, \end{aligned}$$

so the Weyl relations hold in the limit $N \to \infty$, which allows us to call $\lim_{N\to\infty} W_N(\alpha,\beta) = W(\alpha,\beta) = e^{i\alpha q+\beta p}$.

Thus $N \to \infty$ maps our operators into the factorizable, hence commuting product $\mathcal{W} \otimes \mathcal{A}_{\rm loc}$ such that $S_z/N \to -1$, $S_\pm/\sqrt{N} \to (q \mp ip)/\sqrt{2}$. The evolution equations (4.3),(4.4) become (with $\eta_\alpha = e^{i\alpha}\eta$):

$$\dot{\sigma}^k_{\alpha_k} \to 0, \qquad \dot{q} \to p, \qquad \dot{p} \to -q, \qquad \dot{\eta}_\alpha \to i\eta_\alpha, \tag{4.7}$$

$$(\sigma^k_{\alpha_k})' \to 0, \quad q' = \frac{\eta_\alpha - \eta_\alpha^\dagger}{\sqrt{2}}, \quad p' = \frac{\eta_\alpha + \eta_\alpha^\dagger}{\sqrt{2}}, \quad \eta_\alpha' = \frac{q - ip}{\sqrt{2}}\,[\eta_\alpha^\dagger, \eta_\alpha]. \tag{4.8}$$

They are indeed implied by the limiting generators

$$H = (q^2 + p^2 - 1)/2 + \eta_\alpha \eta_\alpha^\dagger, \tag{4.9}$$

$$G_\alpha = \eta_\alpha(q+ip)/\sqrt{2} + \eta_\alpha^\dagger(q-ip)/\sqrt{2}. \tag{4.10}$$

Notice, however, that the evolution of the global operators is not the limit of the (norm) evolution of the local ones [4]. The local σ's remain constant, but the mesoscopic q and p move.

Remark
In the matrix representation

$$\eta_\alpha = \begin{pmatrix} 0 & e^{i\alpha} \\ 0 & 0 \end{pmatrix},$$

the ground state of G_α, $\eta_\alpha^\dagger|0\rangle = (q+ip)|0\rangle = 0$, is given by the vector

$$\left| \begin{matrix} 0 \\ e^{-x^2/2} \end{matrix} \right\rangle.$$

It is thus the same for all α, so there is no symmetry breaking. From the well known relation for the eigenvectors of a total spin $\vec{S}^2$ and S_z, expanding around $s_z = -s$,

$$S_z|s, s_z\rangle = s_z|s, s_z\rangle, \quad S_\pm|s, s_z\rangle = \sqrt{s(s+1) - s_z(s_z \pm 1)}|s, s_z \pm 1\rangle, \tag{4.11}$$

it follows that for all spins in the z-direction the macroscopic S_z/N and the mesoscopic $S_x/\sqrt{N}$, $S_y/\sqrt{N}$ converge.

The ceiling state of H_{SS}

Next we want to examine the ground state of $H_{\mathrm{BCS}} = -H_{\mathrm{SS}}$, which is essentially $S_z^2 - S^2$. Thus S should be as big as possible and $S_z = 0$. Since for each eigenvalue E of H there is an eigenvector to G_α with eigenvalue $\sqrt{E}$, we have to find a maximal eigenvector $|\Omega_\alpha\rangle$ to G_α. This vector will depend on α but all $|\Omega_\alpha\rangle$ are equally suitable for H_{BCS}. Therefore the ground state of H_{BCS} is degenerate.

To get the limiting state we have to find a sequence $|\Omega_{\alpha,N}\rangle$ such that

$$\|(G_\alpha - E_N)\Omega_{\alpha,N}\| = 0. \tag{4.12}$$

With the notation

$$\eta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

we can write the operators G_α and H_{SS} in matrix form as follows:

$$\begin{aligned} G_\alpha &= \begin{pmatrix} 0 & e^{i\alpha}S_+ \\ e^{-i\alpha}S_- & 0 \end{pmatrix}, \\ H_{\mathrm{SS}} &= \begin{pmatrix} S_+S_- & 0 \\ 0 & S_-S_+ \end{pmatrix} = \frac{1}{4}\begin{pmatrix} S^2 - S_z^2 + 2S_z & 0 \\ 0 & S^2 - S_z^2 - 2S_z \end{pmatrix}. \end{aligned} \tag{4.13}$$

An eigenvector has to satisfy

$$\begin{aligned} \begin{pmatrix} 0 & e^{i\alpha}S_+ \\ e^{-i\alpha}S_- & 0 \end{pmatrix} \left| \begin{matrix} \psi_1 \\ \psi_2 \end{matrix} \right\rangle &= \left| \begin{matrix} e^{i\alpha}S_+\,\psi_2 \\ e^{-i\alpha}S_-\,\psi_1 \end{matrix} \right\rangle = \sqrt{E} \left| \begin{matrix} \psi_1 \\ \psi_2 \end{matrix} \right\rangle \\ S_-|\psi_{1,\alpha}\rangle &= e^{i\alpha}\sqrt{E}|\psi_{2,\alpha}\rangle \\ S_+|\psi_{2,\alpha}\rangle &= e^{-i\alpha}\sqrt{E}|\psi_{1,\alpha}\rangle. \end{aligned} \tag{4.14}$$

Evidently, we can choose $|\psi_{1,\alpha}\rangle = |\psi_1\rangle$, $|\psi_{2,\alpha}\rangle = e^{-i\alpha}|\psi_2\rangle$ and all α lead to

$$\begin{aligned}(S^2 - S_z^2 + 2S_z)|\psi_{1,N}\rangle &= 4E_N|\psi_{1,N}\rangle \\ (S^2 - S_z^2 - 2S_z)|\psi_{2,N}\rangle &= 4E_N|\psi_{2,N}\rangle .\end{aligned}$$

With an appropriate symmetrization we need

$$\begin{aligned}|\psi_{1,N}\rangle &= \tfrac{1}{\sqrt{2}}\,|(\uparrow)^{\frac{N}{2}+1}(\downarrow)^{\frac{N}{2}-1}\rangle \\ |\psi_{2,N}\rangle &= \tfrac{1}{\sqrt{2}}\,|(\uparrow)^{\frac{N}{2}}(\downarrow)^{\frac{N}{2}}\rangle \,.\end{aligned}$$

Then

$$\begin{aligned}&(S_z^2 - 2S_z)|\psi_{1,N}\rangle = (4-4)|\psi_{1,N}\rangle = 0 \\ &(S_z^2 + 2S_z)|\psi_{2,N}\rangle = 0.\end{aligned}$$

Thus for finite N we have

$$|\psi_1\rangle = S_+|\psi_2\rangle, \quad |\psi_2\rangle = S_+^{N/2}|0\rangle, \quad 4E_N = N(N+2).$$

This form of $|\psi\rangle$ does not lend itself easily to $N \to \infty$, but we can write $|\psi_2\rangle$ as an integral over vectors with all spins pointing in the $(\cos\alpha, \sin\alpha, 0)$-direction.

Proposition

$$\begin{aligned}|\psi_2\rangle &= \mathcal{C}\int_{-\pi/2}^{\pi/2} d\alpha\, e^{i\alpha S_z}\, e^{i(\pi/2)S_y}|0\rangle =: \mathcal{C}\int_{-\pi/2}^{\pi/2} d\alpha\,|\alpha\rangle \\ \mathcal{C} &= \left(\pi\int_{-\pi/2}^{\pi/2} d\varphi\cos^N\varphi\right)^{-1/2} .\end{aligned} \tag{4.15}$$

Proof

The integrand is periodic with period π; thus, changing $e^{i\alpha S_z}$ to $e^{i(\alpha+\varphi)S_z}$ does not alter the integral. Hence, $e^{i\varphi S_z}|\psi_2\rangle = |\psi_2\rangle$ and $S_z|\psi_2\rangle = 0$. Now

$$\begin{aligned}\langle\psi_2|\psi_2\rangle &= \mathcal{C}^2\int_{-\pi/2}^{\pi/2}\langle\to\ldots\to|e^{-i\varphi' S_z}\,e^{i\varphi S_z}|\to\ldots\to\rangle d\varphi d\varphi' \\ &= \mathcal{C}^2\int_{-\pi/2}^{\pi/2} d\varphi d\varphi'\cos^N(\varphi-\varphi') \;=\; \mathcal{C}^2\pi\int_{-\pi/2}^{\pi/2} d\varphi\cos^N\varphi \;=\; 1.\end{aligned}$$

Coherent superpositions of eigenvectors of the supersymmetry operator G_α lead to the same state in the quasi-local algebra. This state is an integral over product states. The corresponding von Neumann algebra in the GNS representation has a non-trivial centre. The elements of the centre correspond to the orientation in the x–y plane of the pure product states.

Note that although we started with a coherent superposition the limiting state over the local algebra appears to be a mixed state:

$$C^2 \int_{-\pi/2}^{\pi/2} d\alpha d\alpha' \langle \alpha_M | \prod_{k=1}^{M} \sigma^k | \alpha' \rangle = \int_{-\pi/2}^{\pi/2} d\alpha d\alpha' \frac{\cos^{N-M} \alpha'}{\int_{-\pi/2}^{\pi/2} d\varphi d\varphi' \cos^N(\varphi - \varphi')} \langle \alpha_M | \prod_{k=1}^{M} \sigma^k | \alpha'_M \rangle$$

$$\to \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} d\alpha \, \langle \alpha_M | \prod_{k=1}^{M} \sigma^k | \alpha_M \rangle \, .$$

However, we are not interested only in the state over the quasi-local algebra. For global operators we get good convergence properties:

$$\left\| \left(\frac{1}{N} \right)^k \begin{pmatrix} (S_+)^k & 0 \\ 0 & (S_+)^k \end{pmatrix} \left| \begin{matrix} \psi_{1,N} \\ \psi_{2,N} \end{matrix} \right\rangle - \frac{1}{2\sqrt{2}} \left| \begin{matrix} (\uparrow)^{\frac{N}{2}+k} (\downarrow)^{\frac{N}{2}-k} \\ (\uparrow)^{\frac{N}{2}+k+1} (\downarrow)^{\frac{N}{2}-k-1} \end{matrix} \right\rangle \right\| = 0$$

$$\lim \frac{1}{2} \left\langle \begin{matrix} (\uparrow)^{\frac{N}{2}+k} (\downarrow)^{\frac{N}{2}-k} \\ 0 \end{matrix} \right| \begin{pmatrix} S_z & 0 \\ 0 & S_z \end{pmatrix} \left| \begin{matrix} (\uparrow)^{\frac{N}{2}+k} (\downarrow)^{\frac{N}{2}-k} \\ 0 \end{matrix} \right\rangle = 2k.$$

We can therefore interpret the limits of the expectation values as expectation values of operators acting in the Hilbert space $L^2[T, d\varphi) \otimes C^2$, with

$$\begin{array}{lllllll} \eta & \longrightarrow & 1 \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \qquad & \dfrac{S_+}{N} & \longrightarrow & e^{2\pi i \varphi} \\ S_z & \longrightarrow & 2p_\varphi & & \dfrac{S_-}{N} & \longrightarrow & e^{-2\pi i \varphi}, \end{array} \tag{4.16}$$

where p_φ is the momentum on T with periodic boundary conditions.

5 Comparison with the BCS theory

Without referring to supersymmetry, we consider the Hamiltonian

$$H'_{\mathrm{BCS}} = -\frac{S_+ S_-}{N} = -\frac{S^2 - S_z^2}{4N} = -\frac{S_x^2 + S_y^2}{4N} \, . \tag{5.1}$$

This operator differs from the previous one only by S_z/N, which is a bounded operator and therefore converges to an operator in the centre, which does not affect the dynamics. Thus the two Hamiltonians can be thought of as describing one and the same physical situation.

In this case the ground-state energy can be approximated by the rotated ground state of H_{SS}, namely by

$$|\psi_{\alpha,N}\rangle = \prod \left(\frac{1}{\sqrt{2}} \right)^N \left| \begin{matrix} e^{i\alpha} \\ e^{-i\alpha} \end{matrix} \right\rangle . \tag{5.2}$$

For the limits of the expectations of the local operators we obtain

$$\lim\langle\psi_{\alpha,N}|\prod\sigma^{i_j}_{\alpha_j}|\psi_{\alpha,N}\rangle = \prod_j\left(\frac{1}{2}\right)^N\left\langle\begin{matrix}e^{i\alpha}\\ e^{-i\alpha}\end{matrix}\middle|\sigma^{ij}_{\alpha_j}\middle|\begin{matrix}e^{i\alpha}\\ e^{-i\alpha}\end{matrix}\right\rangle, \tag{5.3}$$

which is a pure product state.

To evaluate the global operators in this scaling limit, we must adjust the renormalization to the parameter α. In particular, for $\alpha = 0$:

$$\begin{aligned}
&\lim\sum_k\frac{\sigma^k_x}{N} = 1\\
&\lim\langle\psi_{0,N}|A_{\rm loc}\left(\frac{\sum(\sigma^k_x-1)}{\sqrt{N}}\right)^2 A_{\rm loc}|\psi_{0,N}\rangle = 0\\
&\lim\langle\psi_{0,N}|A_{\rm loc}\,e^{ir\sum(\sigma^k_x-1)/\sqrt{N}}\,A_{\rm loc}|\psi_{0,N}\rangle = 1\,.
\end{aligned}$$

For σ_y and σ_z we get

$$\begin{aligned}
&\lim\langle\psi_{0,N}|\frac{\sum\sigma^k_y}{\sqrt{N}}|\psi_{0,N}\rangle = 0\\
&\lim\langle\psi_{0,N}|\frac{(\sum\sigma^k_y)^2}{N}|\psi_{0,N}\rangle = 1\\
&\lim\langle\psi_{0,N}|e^{ir\sum\sigma^k_y/\sqrt{N}}|\psi_{0,N}\rangle = e^{-\frac{r^2}{2}}\\
&\lim\langle\psi_{0,N}|e^{ir\sum\sigma^k_z/\sqrt{N}}|\psi_{0,N}\rangle = e^{-\frac{r^2}{2}}\,.
\end{aligned}$$

Similarly to the ground state of $H_{\rm SS}$, we can interpret

$$\begin{aligned}
&\lim e^{ir\sum\sigma^k_y/\sqrt{N}} = e^{irq}\\
&\lim e^{is\sum\sigma^k_z/\sqrt{N}} = e^{isp}\\
&\lim e^{ir\sum\sigma^k_y/\sqrt{N}}e^{is\sum\sigma^k_x/\sqrt{N}}e^{-ir\sum\sigma^k_y/\sqrt{N}} = e^{irs}e^{isp}\,,
\end{aligned} \tag{5.4}$$

as a non-trivial global algebra, so that e^{irq} and e^{isp} satisfy the Weyl relations in $L^2(R,dq)$, and q and p can be interpreted as space and momentum operators respectively. This is nothing else but the fluctuation algebra as discussed in [3].

The time evolution however shows new features. On the quasi-local algebra it corresponds to a rotation of the individual spins around the x-axis [6]. For the global algebra we get something quite different. In leading order in N we have

$$\left[\frac{S_+S_-}{N},\frac{S_x-N}{\sqrt{N}}\right] = -i\frac{S_yS_z}{2N\sqrt{N}},\quad \left[\frac{S_+S_-}{N},\frac{S_y}{\sqrt{N}}\right] = i\frac{S_xS_z}{2N\sqrt{N}},\quad \left[\frac{S_+S_-}{N},\frac{S_z}{\sqrt{N}}\right] = 0 \tag{5.5}$$

This corresponds, for the Weyl algebra, to

$$[H,q] = -ip, \qquad [H,p] = 0,$$

and thus to a free evolution, $q \to q + pt$, $p = \text{const}$. Therefore no invariant state exists and the state over the fluctuation algebra has to change in time.

Of course also in this setting we can construct a coherent superposition of the eigenvectors

$$|\psi_{g,N}\rangle = \left| \int d\alpha \, g_{\alpha,N} \psi_{\alpha,N} \right\rangle .$$

With the appropriate care in the normalization, we obtain on the local level

$$\lim \left\langle \int d\alpha \, g_{\alpha,N} \psi_{\alpha,N} | A_{\text{loc}} | \int d\beta \, g_{\beta,N} \psi_{\beta,N} \right\rangle = \int |g_\alpha|^2 d\alpha \langle \psi_{\alpha,N} | A_{\text{loc}} | \psi_{\alpha,N} \rangle$$

because different $\psi_{\alpha,N}$ lead to inequivalent representations, i.e.

$$\lim \langle \psi_{\alpha,N} | \psi_{\beta,N} \rangle = 0, \qquad \alpha \neq \beta \, .$$

The coherent superposition escapes our observation on the local level, where it gives the same expectation value as the incoherent superposition. If we move to the mesoscopic level, then the fluctuation algebra has no transparent interpretation for the statistical superposition of the states. For the coherent superposition, especially for $g = 1$, we recover the ceiling state

$$|\psi_{1,N}\rangle = \int_{-\pi}^{\pi} d\alpha \, |\psi_{\alpha,N}\rangle, \tag{5.6}$$

and the global algebra changes from $L^2(R, dq)$ to $L^2(T, dq)$, where the global operators are obtained in a different scaling.[1]

6 Summary

We have studied three sets of operators — local, mesoscopic and macroscopic — in representations based on different states of H_{SS}: the ground state (GS), the Bogoliubov state in the x-direction (BS), and the ceiling state (CS). Recall the limiting algebras in these three cases:

$$\begin{array}{ll}
\text{GS} & \left\{ \begin{array}{l} \left(\dfrac{S_x}{2\sqrt{N}}, \dfrac{S_y}{2\sqrt{N}}, \dfrac{S_z + N}{\sqrt{N}} \right) \longrightarrow (q, -p, 0) \\ \left(\dfrac{S_x}{N}, \dfrac{S_y}{N}, \dfrac{S_z}{N} \right) \longrightarrow (0, 0, -1) \end{array} \right. \\
\text{BS} & \left\{ \begin{array}{l} \left(\dfrac{S_x - N}{\sqrt{N}}, \dfrac{S_y}{2\sqrt{N}}, \dfrac{S_z}{2\sqrt{N}} \right) \longrightarrow (0, -q, p) \\ \left(\dfrac{S_x}{N}, \dfrac{S_y}{N}, \dfrac{S_z}{N} \right) \longrightarrow (1, 0, 0) \end{array} \right. \\
\text{CS} & \left\{ \begin{array}{l} \text{mesoscopic operators do not converge} \\ \text{macroscopic } \dfrac{S_\pm}{N} \to e^{\pm i\varphi}, \quad S_z \to 2p_\varphi. \end{array} \right.
\end{array} \tag{6.1}$$

[1]Note that a different scaling is necessary for the weak decay properties [7], but here we have a non-trivial centre and thus extend the Abelian algebra of the centre to an irreducible algebra over the centre in which all quantum effects are contained.

In Tables 1 and 2 our confusing results for the time evolution and for the supertransformation respectively, for this variety of operators are collected.

	Local	Mesoscopic	Macroscopic
GS	$\overrightarrow{\sigma}^{(j)}(t) = \overrightarrow{\sigma}^{(j)}$	$q(t) + ip(t) = e^{it}(q + ip)$ $\eta(t) = e^{it}\eta$	constant
BS	$\sigma_x^j(t) = \sigma_x^j(0)$ $\sigma_y^j(t) + i\sigma_z^j(t) = e^{it}(\sigma_y^j + i\sigma_z^j)$	$q(t) = q - pt, \quad p(t) = p$ $\eta(t) = \eta$	constant
CS	$\overrightarrow{\sigma}$ rotates around $(\cos\varphi, \sin\varphi, 0)$	p_φ, φ constant	constant

Table 1: The time evolution

	Local	Mesoscopic	Macroscopic
GS	$\overrightarrow{\sigma}^{(j)}(s) = \overrightarrow{\sigma}^{(j)}(0)$	$q' + ip' = i\eta^\dagger$ $\eta' = i(q - ip)[\eta, \eta^\dagger]$	$\overrightarrow{S}(s) =$ constant
BS	$\overrightarrow{\sigma}^{(j)}(s) = \overrightarrow{\sigma}^{(j)}(0)$	$\eta' \sim \sqrt{N}$ becomes infinite	$\overrightarrow{S}(s) =$ constant
CS	σ' is ill-defined		S'_z becomes infinite

Table 2: The supertransformation

We have seen that the ceiling state ω_c of H_{SS} and the Bogoliubov state $\Omega_\alpha = \langle\Psi_\alpha| \, . \, |\Psi_\alpha\rangle$ lead to the same energy per particle. In the BCS theory, there is some discussion [5], which one is better. We collect a few arguements to compare the ensuing representations π_c and π_α.

(i) ω_c satisfies ODLRO (off-diagonal long-range order), ω_α does not: for $k \neq j$

$$|\omega_c(\sigma_x^k \sigma_x^j) - \omega_c(\sigma_x^k)\omega_c(\sigma_x^j)| = 1/2 \tag{6.2}$$

$$|\omega_\alpha(\sigma_x^k \sigma_x^j) - \omega_\alpha(\sigma_x^k)\omega_c\alpha(\sigma_x^j)| = 0; \tag{6.3}$$

(ii) In π_c the time evolution mixes local and mesoscopic quantities, in π_α it is strictly local and corresponds to a rotation around the α-axis;

(iii) In π_α the Josephson phase is fixed to be α, in π_c it is a dynamical variable;

(iv) π_α represents the quasi-local variables irreducibly, in π_c the weak closure contains the non-trivial commuting macroscopic observables.

Remarks:

1. ODLRO is the basis of the Meissner effect [8]. The spectrum of p_φ corresponds to the quantization of the magnetic flux;

2. The absolute phase α has no physical meaning. What can be measured is the phase difference between superconductors. This means either staying in the mesoscopic algebra of π_c or comparing the inequivalent representations π_α and $\pi_{\alpha'}$.

References

[1] P. Fendley, K. Schoutens, J. de Boer, *Phys. Rev. Lett.* **90** (2003) 120402;
P. Fendley, B. Nienhuis, K. Schoutens, *J. Phys. A* **36** (2003) 12399–12424.

[2] E. Witten, *Nucl. Phys. B* **185** (1981) 513.

[3] D. Goderis, A. Verbeure, P. Vets, *Prob. Th. Rel. Fields*, **82** (1989) 527; *Commun. Math. Phys.* **128** (1990) 533; *Il Nuovo Cim.* **106B** (1991) 375.

[4] H. Narnhofer, *Found. Phys. Lett.* **17** (2004)235–255.

[5] Ph.A. Martin, F. Rothen, *Many-Body Problems and Quantum Field Theory. An Introduction*, Second Ed. (Springer Verlag, Berlin Heidelberg, 2004).

[6] W. Thirring, A. Wehrl, *Commun. Math. Phys.* **4** (1967) 303

[7] A. Verbeure, V.A. Zagrebnov, *J. Stat. Phys.* **69** (1992) 329

[8] G. Sewell, *Quantum Mechanics and its Emergent Macrophysics* (Princeton Univ. Press, Princeton and Oxford, 2002).